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On Convergence of Convex Sets and Relative Chebyshev Centers

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1. INTRODUCTION

Let X be a normed linear space. Given a closed and bounded subset $F(\neq \emptyset)$ of X, let $r(F, X) \equiv \sup\{\|x-z\| : z \in F\}$ denote the radius of the smallest closed ball centered at x, covering F. For a nonempty subset A of X, let

$$\operatorname{rad}(F; A) \equiv \inf\{r(F, x) : x \in A\}$$

denote the *Chebyshev radius* of F in A. Any point $x \in A$ for which $r(F, x) = \operatorname{rad}(F; A)$ is called a *relative Chebyshev center of F in A*, and the (possibly void) set of relative Chebyshev centers of F in A is denoted by $\operatorname{Cent}(F; A)$. In terms of applications, we may view F as some data set, and elements of $\operatorname{Cent}(F; A)$ as best representing the data set in A. In the sequel we shall refer to F as the *data set*, and to A as the *constraint set*.

The study of relative Chebyshev centers (also called *best simultaneous approximations*), initiated by A. L. Garkavi [22] almost 25 years ago, has drawn more attention during the last decade. Questions concerning the

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existence, uniqueness, and stability of relative centers have been explored by several authors (cf., e.g., [1, 2, 3, 23, 24, 28, 29, 36]). For a recent survey of results in this direction, the reader may consult [4] (cf. also the expository article [21]).

Continuity properties of the set-valued mapping $F \to \text{Cent}(F; A)$, where the closed and bounded subsets of X are equipped with the usual Hausdorff metric topology, have also been well-studied [11, 30, 34, 35]. Here we look at the relative center mapping as a *bivariate* set-valued function, where the constraint variable ranges over convex sets, appropriately topologized. Although the full strength of the Hausdorff metric topology on the data set coordinate space is required to obtain results, a much weaker topology suffices for the constraint set coordinate space. The correct path to follow is indicated by recent research on metric projections as bivariate functions, or at least as functions of the set variable, and Mosco convergence [5, 9, 37, 38]; the metric projection map is a particular case of the relative center map, where the data set is a singleton. Although our more general results assert only weak upper semicontinuity, when either X is finite dimensional and rotund, or both X and X^* have Fréchet differentiable norms except at the origin, then the convergence of a net $\langle A_\lambda \rangle$ to A in a topology compatible with Mosco convergence is actually equivalent to the norm convergence of the net $\langle \text{Cent}(A_\lambda; F) \rangle$ to $\langle \text{Cent}(A; F) \rangle$ for each closed and bounded set F admitting farthest points. In the case X is a separable reflexive space, we also obtain a subtle, rather surprising, generic theorem on points of single valuedness of the restricted center map. Finally, we investigate the weakest topology on convex sets such that $A \to \text{rad}(F; A)$ is continuous for each closed and bounded set F.

2. Notation and Terminology

In the sequel, X will be a normed linear space and X^* will denote its normed dual. The origin and closed unit ball of X (resp. X^*) will be denoted by θ and U (resp. θ^* and U^*). Also, S (resp. S^*) will denote the unit sphere (norm one elements) of X (resp. X^*). If a net $\langle x_\lambda \rangle$ converges weakly (resp. weak*) to x, then we write $x = w\text{-lim}_\lambda x_\lambda$ (resp. $x = w^*\text{-lim}_\lambda x_\lambda$). Norm convergence will be simply denoted by $x = \lim_\lambda x_\lambda$.

We distinguish the following classes of normed spaces:

$(Rf) \equiv$ the reflexive Banach spaces,

$(R) \equiv$ the rotund (strictly convex) normed spaces,

$(R^*) \equiv$ the normed spaces whose duals are in (R),

$(H) \equiv$ the normed spaces for which weak convergence of a net in S to a point of S implies norm convergence,

$(H^*) \equiv$ the normed spaces for which weak* convergence of a net in S^* to a point of S^* implies norm convergence,

$(F) \equiv$ the normed spaces whose norms are Fréchet differentiable except at the origin,

$(F^*) \equiv$ the normed spaces whose dual norms are Fréchet differentiable except at the origin.

Ordinarily, the definitions of the classes (H) and (H^*) are given in terms of limits of sequences rather than nets (cf. [25, 37]). However, there seems to be little loss of generality in our more restrictive definition, for the most important spaces in (H) or (H^*) in the usual sense satisfy the more restrictive definition as well. Besides, we need the full strength of our definition to deal with net arguments which arise in the consideration of certain topologies on sets of convex subsets that fail to be first countable. It is well-known ([39], or [25, pp. 147–149]) that $X \in (Rf) \cap (R) \cap (H)$ if and only if $X \in (F^*)$, so that

$$(F) \cap (F^*) = (Rf) \cap (R) \cap (H) \cap (R^*) \cap (H^*).$$

Apparently, the class $(Rf) \cap (R) \cap (H)$ was introduced for the first time by Fan and Glicksberg [18].

We also distinguish the following classes of subsets of X as well:

$CL(X) =$ the nonempty closed subsets of X,

$CB(X) =$ the nonempty closed and bounded subsets of X,

$CC(X) =$ the nonempty closed and convex subsets of X,

$C^*C(X^*) =$ the nonempty weak*-closed and convex subsets of X^*.

Recall that the *Hausdorff distance* H between nonempty closed subsets A and B of X is defined by the formula

$$H(A, B) = \inf\{\alpha : A + \alpha U \supset B \text{ and } B + \alpha U \supset A\}.$$

Hausdorff distance so defined yields an infinite valued metric on $CL(X)$, which is complete when X is complete [26, p. 44]. Restricted to $CB(X)$, a closed subset of $CL(X)$ [26, p. 45], it defines a complete finite valued metric, when X is complete. We will denote the topology of Hausdorff distance by τ_H.

We now turn to Mosco convergence (due to Mosco [31]), which has become the convergence notion of choice for convex analysts working in reflexive spaces, in view of its stability with respect to duality [6, 10, 32]. A sequence $\langle A_n \rangle$ in $CC(X)$ where X is a reflexive space is declared *Mosco convergent* to a convex set A in $CC(X)$ provided

(i) at each a in A, there exists a sequence $\langle a_n \rangle$ strongly convergent to a such that for each n, $a_n \in A_n$, and

(ii) whenever $\langle n(k) \rangle$ is an increasing sequence of positive integers and $a_{n(k)} \in A_{n(k)}$ for each k, then the weak convergence of $\langle a_{n(k)} \rangle$ to $x \in X$ implies $x \in A$.

Evidently, Mosco convergence is much weaker than Hausdorff metric convergence, for the sequence of lines $\langle \{(x, y): y = x/n\} \rangle$ is Mosco convergent to the line $y = 0$. The basic source of information on Mosco convergence of sequences of sets remains the comprehensive thesis of Sonntag [37].

In [9], a "hit-and-miss" (Vietoris-type) topology τ_M compatible with Mosco convergence of sequences was introduced, called the *Mosco topology* therein (for a more analytical approach to topologizing Mosco convergence, consult [6]). In terms of the standard plus and minus notation for hyperspaces, the Mosco topology τ_M is generated by all sets of the form

$$V^- \equiv \{A \in CC(X): A \cap V \neq \varnothing\}$$

$$(K^c)^+ \equiv \{A \in CC(X): A \cap K = \varnothing\},$$

where V is a norm open of X and K is a weakly compact subset of X. The topology may be described as a weak topology as follows: it is the weakest topology on $CC(X)$ such that for each fixed weakly compact set K, the *gap functional* $A \to \inf\{\|a - k\|: a \in A$ and $k \in K\}$ is continuous on $CC(X)$ [9, Theorem 3.3]. In particular, if a net $\langle A_\lambda \rangle$ in $CC(X)$ is τ_M-convergent to A, then $\langle d(\cdot, A_\lambda) \rangle$ must converge pointwise to $d(\cdot, A)$. In the literature, pointwise convergence of distance functions is often called *Wijsman convergence* [40]. In a reflexive space, the topology of Wijsman convergence for closed convex sets, i.e., the weakest topology on $CC(X)$ for which the functionals $A \to d(x, A)$ are each continuous, may also be described as a "hit-and-miss" topology (see [9, Theorem 3.5] and Section 5 below).

With respect to topological properties and metrizability, the Mosco topology in a reflexive space is always Hausdorff and completely regular [9, Theorem 3.4], and it is first countable if and only if X is separable. Moreover, if X is separable then $\langle CC(X), \tau_M \rangle$ is actually separable and completely metrizable [9, Section 4]. Its stability with respect to duality is expressed in terms of the continuity of the polar map, or in terms of the continuity of the Young–Fenchel transform for proper lower semicontinuous convex functions, identified with their epigraphs [10].

In this article, we call set-valued functions multifunctions. By a *multifunction* Γ from a topological space T to a topological space Y, we mean a function from T to $CL(Y)$. A multifunction Γ is said to be *upper semicontinuous* (abbreviated usc) [26] if for each open subset V of Y, the set $\{t \in T: \Gamma(t) \subset V\}$ is open in T. Equivalently, Γ is usc if for each closed subset E of Y, the set

$$\Gamma^{-1}(E) \equiv \{t \in T: \Gamma(t) \cap E \neq \emptyset\}$$

is closed in T. If, in addition, the values of Γ are compact subsets of Y, Γ is called an *usco map* [14]. If X (resp. X^*) is a normed space (resp. a dual normed space) equipped with the weak topology (resp. weak* – topology), then we employ the terms w-usc, w-usco (resp. w^*-usc, w^*-usco), for usc and usco maps into X so topologized. In particular, if X is a reflexive Banach space, the *metric projection* multifunction from $X \times CC(X)$ to X, defined by

$$(x, A) \rightarrow \{a \in A: \|a - x\| = d(x, A)\}$$

is w-usco, if we equip X with the norm topology and $CC(X)$ with the Mosco topology [9, Theorem 5.1]. As one main result, we present an analagous theorem for the relative center map—*valid in dual spaces*—that completely subsumes the above metric projection continuity result. This requires a generalization of the Mosco topology to dual spaces, to be undertaken in Section 3.

In the sequel we shall need to consider a certain multifunction J from X to X^*, defined by

$$J(x) = \{y \in X^*: \langle x, y \rangle = \|x\|^2 = \|y\|^2\}.$$

This mapping has nonempty w^*-compact convex subsets of X^* as values, and is usually called the *duality mapping* [13, 15, 27, 37]. If $X \in (R^*)$, or, more generally, if X is smooth [25, p. 106], then J is single valued and norm–w^* continuous. If, in addition, X is also in (H^*), then J is norm–norm continuous. If $X \in (Rf)$ and $X \in (R^*)$, then J is surjective, and if, in addition, $X \in (R)$, then J is also injective. Thus, if $X \in (F) \cap (F^*)$, then J^{-1} exists and both J and J^{-1} are norm–norm continuous.

3. Upper Semicontinuity of the Relative Center Mapping

In this section, we establish basic upper semicontinuity properties of the relative center mapping as a bivariate multifunction. Not unexpectedly, our efforts will require continuity results for the *radius function* $(F, A) \rightarrow \text{rad}(F; A)$. First, however, we find it convenient to recall some of the known results on the upper semicontinuity of $F \rightarrow \text{Cent}(F; A)$. Our first lemma is simple yet fundamental in all that follows. The proof is left to the reader.

LEMMA 3.1. *Let X be a normed linear space, and let $\{F, G\} \subset CB(X)$. For each x and z in X, we have*

$$|r(F, x) - r(F, z)| \leqslant \|x - z\| \qquad \text{and} \qquad |r(F, x) - r(G, x)| \leqslant H(F, G).$$

Moreover, if A is any nonempty subset of X, then

$$|\text{rad}(F; A) - \text{rad}(G; A)| \leqslant H(F, G).$$

As an immediate consequence of Lemma 3.1, the functional $x \to r(F, x)$ is norm continuous on X, and the functionals $F \to r(F; x)$ and $F \to \text{rad}(F; A)$ on $\langle CB(X), \tau_H \rangle$ are continuous.

DEFINITION. A subset A of X (resp. X^*) is said to be *cent-compact* (resp. *w^*-cent compact*) for $CB(X)$ (resp. $CB(X^*)$) if for each $F \in CB(X)$ (resp. $CB(X^*)$), each net $\langle x_\lambda \rangle$ in A satisfying $\text{rad}(F; A) = \lim_\lambda r(F, x_\lambda)$ has a convergent (resp. w^*-convergent) subnet to a point of A.

PROPOSITION 3.2. *If A is cent-compact (resp. w^*-cent-compact) for $CB(X)$ (resp. $CB(X^*)$), then for each F in $CB(X)$ (resp. in $CB(X^*)$), $\text{Cent}(F; A)$ is nonempty, and the relative center map $F \to \text{Cent}(F; A)$ is usco (resp. w^*-usco) on $CB(X)$ (resp. $CB(X^*)$), equipped with τ_H.*

Proof. Let F be in $CB(X)$ (resp. in $CB(X^*)$), and let A be cent-compact (resp. w^*-cent-compact). The proof of nonemptiness for $\text{Cent}(F; A)$ follows from the well-known existence principle of Garkavi [22] (cf. also [2, Proposition A]). Evidently, since A is cent-compact (resp. w^*-cent-compact), $\text{Cent}(F; A)$ is compact (resp. w^*-compact). It remains therefore to prove that $F \to \text{Cent}(F; A)$ is usc (resp. w^*-usc). For this, we must show

$$\text{Cent}^{-1}(E) \equiv \{F \in CB(X) \text{ (resp. } CB(X^*)): \text{Cent}(F; A) \cap E \neq \varnothing\}$$

is τ_H-closed for each fixed closed (resp. w^*-closed) subset E of X (resp. X^*). Let $\langle F_\lambda \rangle$ be a net in $\text{Cent}^{-1}(E)$ such that $F = \tau_H\text{-lim}\, F_\lambda$. It suffices to prove that $F \in \text{Cent}^{-1}(E)$. For each λ, choose $a_\lambda \in \text{Cent}(F_\lambda; A) \cap E$. By definition, $r(F_\lambda, a_\lambda) = \text{rad}(F_\lambda; A)$; so, by Lemma 3.1,

$$\begin{aligned} |\text{rad}(F; A) - r(F, a_\lambda)| &\leqslant |\text{rad}(F; A) - \text{rad}(F_\lambda; A)| + |r(F_\lambda, a_\lambda) - r(F, a_\lambda)| \\ &\leqslant 2 \cdot H(F, F_\lambda). \end{aligned}$$

As a result, $r(F, a_\lambda) \to \text{rad}(F; A)$. Since A is cent-compact (resp. w^*-cent-compact), $\langle a_\lambda \rangle$ has a convergent (resp. a w^*-convergent) subnet $\langle a_\mu \rangle$

convergent (resp. w^*-convergent) to a point $a_0 \in A \cap E$. Now $a \to r(F, a)$ is norm continuous if $A \subset X$, and if $A \subset X^*$, then it is weak*-lower semicontinuous by the weak*-lower semicontinuity of the dual norm. In either case, we have

$$r(F, a_0) \leqslant \liminf_\mu r(F, a_\mu) = \operatorname{rad}(F; A).$$

Thus, $a_0 \in \operatorname{Cent}(F; A) \cap E$ so that $F \in \operatorname{Cent}^{-1}(E)$. We conclude that $\operatorname{Cent}^{-1}(E)$ is τ_H-closed. ∎

The preceding proposition contains these well-known [4] special cases: The relative center map $F \to \operatorname{Cent}(F; A)$ is

(1) usco if A is closed and X is finite dimensional (or more generally, boundedly compact);

(2) w-usco if A is a w-closed subset of a reflexive Banach space X;

(3) w^*-usco if A is a w^*-closed subset of the dual X^* of a normed space X.

We now explore the upper semicontinuity of the relative center-map regarded as a multifunction of both the arguments F and A. Although we could develop our theory in reflexive spaces using the Mosco topology on the constraint set coordinate space, it is no harder to work in the more general setting of a dual normed space X^*, provided we modify the Mosco topology appropriately.

DEFINITION. Let X^* be a dual normed space. The *dual Mosco topology* τ_{M^*} on the weak* closed convex subsets $C^*C(X^*)$ of X^* is generated by all sets of the form

$$V^- \equiv \{A \in C^*C(X^*): A \cap V \neq \varnothing\}$$

$$(K^c)^+ \equiv \{A \in C^*C(X^*): A \cap K = \varnothing\},$$

where V is a norm open subset of X^* and K is w^*-compact subset of X^*.

Since a reflexive space X can be regarded as the dual of X^*, and since X is reflexive if and only if X^* is reflexive, the weak and weak* topologies on X coincide, provided X is reflexive. In view of this, for reflexive X, the two topologies τ_M and τ_{M^*} coincide on $CC(X)$. It is easy to see that many of the basic facts about $\langle CC(X), \tau_M \rangle$ for reflexive spaces established in [9], e.g., Theorems 3.3–3.5, remain valid for $\langle C^*C(X^*), \tau_{M^*} \rangle$ for X an arbitrary normed space, with the obvious modifications.

LEMMA 3.3. *Let X^* be a dual normed space, and let $F \in CB(X^*)$ be fixed. Then the radius functional $A \to \operatorname{rad}(F; A)$ on $C^*C(X^*)$ equipped with τ_{M^*} is continuous.*

Proof. Fix $A_0 \in C^*C(X^*)$, and pick up $a_0 \in \text{Cent}(F; A_0)$ (which is nonempty by Proposition 3.2). Then $\{y \in X^* : \|y - a_0\| < \varepsilon\}^-$ is a τ_{M^*}-neighborhood of A_0. Suppose $A \in \{y \in X^* : \|y - a_0\| < \varepsilon\}^-$; then there is $a \in A$ with $\|a - a_0\| < \varepsilon$. By Lemma 3.1 we have

$$\text{rad}(F; A) \leqslant r(F, a) \leqslant r(F, a_0) + \varepsilon = \text{rad}(F; A_0) + \varepsilon,$$

which proves that $A \to \text{rad}(F; A)$ is upper semicontinuous at $A = A_0$. Lower semicontinuity of the functional holds trivially if $\text{rad}(F; A_0) = \text{rad}(F; X^*)$. Otherwise, let β be an arbitrary number with $\text{rad}(F; X^*) < \beta < \text{rad}(F; A_0)$. Consider the weak*-compact set $K \equiv \bigcap_{y \in F} y + \beta U^*$. Since $\beta > \text{rad}(F; X^*)$, the set K is nonempty, and since $\beta < \text{rad}(F; A_0)$, K and A_0 are disjoint. Furthermore, if $A \in (K^c)^+$, then $\text{rad}(F; A) > \beta$. This proves τ_{M^*}-lower semicontinuity of $A \to \text{rad}(F; A)$ at $A = A_0$. ∎

Letting F run over the singleton subsets of X^*, it follows immediately from Lemma 3.3 that the τ_{M^*}-convergence of a net $\langle A_\lambda \rangle$ in $C^*C(X^*)$ to $A \in C^*C(X^*)$ entails the pointwise convergence of $\langle d(\cdot, A_\lambda) \rangle$ to $d(\cdot, A)$ (see [9, Theorem 3.5]).

LEMMA 3.4. *Let X^* be a dual normed space. Let $CB(X^*)$ be equipped with the topology τ_H and $C^*C(X^*)$ be equipped with the topology τ_{M^*}. Then the functional $(F, A) \to \text{rad}(F; A)$ on $CB(X^*) \times C^*C(X^*)$ is continuous.*

Proof. Consider a net $\langle (F_\lambda, A_\lambda) \rangle_{\lambda \in \Lambda}$ in $\langle CB(X^*), \tau_H \rangle \times \langle C^*C(X^*), \tau_{M^*} \rangle$ convergent to (F, A). Again applying Lemma 3.1, we have

$$\begin{aligned} &|\text{rad}(F_\lambda; A_\lambda) - \text{rad}(F; A)| \\ &\quad \leqslant |\text{rad}(F_\lambda; A_\lambda) - \text{rad}(F; A_\lambda)| + |\text{rad}(F; A_\lambda) - \text{rad}(F; A)| \\ &\quad \leqslant H(F_\lambda, F) + |\text{rad}(F; A_\lambda) - \text{rad}(F; A)| \end{aligned}$$

and continuity follows from Lemma 3.3. ∎

LEMMA 3.5. *Let X^* be a dual normed space. Let $\langle A_\lambda \rangle_{\lambda \in \Lambda}$ be a net in $C^*C(X^*)$ with $A = \tau_{M^*}\text{-lim}\, A_\lambda$. Suppose for each index λ, $a_\lambda \in A_\lambda$, and $\langle a_\lambda \rangle$ is eventually norm bounded. Then A contains each w^*-cluster point of $\langle a_\lambda \rangle$.*

Proof. Let z be an arbitrary w^*-cluster point of $\langle a_\lambda \rangle$. By assumption, for some $\mu \in \Lambda$ and $\alpha > 0$, we have $\{a_\lambda : \lambda \geqslant \mu\} \subset \alpha U^*$. Suppose that $z \notin A$. By the separation theorem [17, p. 417], there exists $x_0 \in X$ and $\beta \in R$ with $\sup\{\langle x_0, a \rangle : a \in A\} < \beta < \langle x_0, z \rangle$. Consider this w^*-compact set:

$$K \equiv \alpha U^* \cap \{y \in X^* : \langle x_0, y \rangle \geqslant \beta\}.$$

Since $\{y \in X^*: \langle x_0, y \rangle \geqslant \beta\}$ is a w^*-neighborhood of z, the choice of α ensures that K meets $\langle A_\lambda \rangle$ frequently, whereas $K \cap A = \varnothing$. This violates $A = \tau_{M^*}$-lim A_λ, and we conclude that $z \in A$ must hold. ∎

We now come to one of our main results.

THEOREM 3.6. *Let X^* be a dual normed space. Then the relative center map* Cent: $\langle CB(X^*), \tau_H \rangle \times \langle C^*C(X^*), \tau_{M^*} \rangle \to C^*C(X^*)$ *is w^*-usco.*

Proof. Obviously, the values of the relative center map are w^*-compact convex sets. To prove that Cent is w^*-usc, it suffices to show that

$$\mathrm{Cent}^{-1}(E) \equiv \{(F, A) \in CB(X^*) \times C^*C(X^*): \mathrm{Cent}(F; A) \cap E \neq \varnothing\}$$

is closed in $\langle CB(X^*), \tau_H \rangle \times \langle C^*C(X^*), \tau_{M^*} \rangle$ for any w^*-closed subset E of X^*. To this end, let $\langle (F_\lambda, A_\lambda) \rangle_{\lambda \in \Lambda}$ be a net in $\mathrm{Cent}^{-1}(E)$ convergent to (F, A). This means that $F = \tau_H$-lim F_λ and $A = \tau_{M^*}$-lim A_λ. Since $\mathrm{Cent}(F_\lambda; A_\lambda) \cap E \neq \varnothing$, we can choose for each index λ a point a_λ in $\mathrm{Cent}(F_\lambda; A_\lambda) \cap E$. Now for all λ sufficiently large, $H(F_\lambda, F) < 1$, and by Lemma 3.4, $|\mathrm{rad}(F_\lambda; A_\lambda) - \mathrm{rad}(F; A)| < 1$. As a result, with $\alpha_\lambda = \mathrm{rad}(F_\lambda; A_\lambda)$ and $\alpha = \mathrm{rad}(F; A)$, we have for all λ sufficiently large,

$$\begin{aligned}\mathrm{Cent}(F_\lambda, A_\lambda) = A_\lambda \cap \bigcap \{y + \alpha_\lambda U^*: y \in F_\lambda\} &\subset \bigcup \{y + \alpha_\lambda U^*: y \in F_\lambda\} \\ &\subset F + (\alpha + 2)\, U^*.\end{aligned}$$

Since F is bounded, it follows that for some $\mu \in \Lambda$, $\{a_\lambda: \lambda \geqslant \mu\}$ is bounded. By weak*-compactness of closed balls in X^*, we may assume by passing to a subnet that $\langle a_\lambda \rangle$ weak*-converges to some $z \in E$. Clearly, z must lie in A, by virtue of Lemma 3.5. We now claim that $z \in \mathrm{Cent}(F; A)$.

Let $\varepsilon > 0$ be arbitrary, and fix $y \in F$ with $\|z - y\| > r(F, z) - \varepsilon$. By the weak*-lower semicontinuity of the norm in X^* and by Lemma 3.4, we have for all sufficiently large λ, (i) $\|a_\lambda - y\| > \|z - y\| - \varepsilon$; (ii) $H(F_\lambda, F) < \varepsilon$; (iii) $|\mathrm{rad}(F_\lambda; A_\lambda) - \mathrm{rad}(F; A)| < \varepsilon$. For all such λ we obtain

$$\begin{aligned}r(F, z) &< \|z - y\| + \varepsilon \leqslant \|a_\lambda - y\| + 2\varepsilon \leqslant r(F, a_\lambda) + 2\varepsilon \\ &\leqslant r(F_\lambda, a_\lambda) + 3\varepsilon = \mathrm{rad}(F_\lambda; A_\lambda) + 3\varepsilon < \mathrm{rad}(F; A) + 4\varepsilon.\end{aligned}$$

Since ε was arbitrary, this proves $r(F, z) \leqslant \mathrm{rad}(F; A)$, so that $z \in \mathrm{Cent}(F; A)$. We conclude that $(F, A) \in \mathrm{Cent}^{-1}(E)$, completing the proof that the relative center map is w^*-usco. ∎

COROLLARY 3.7. *Let X be a reflexive Banach space. If $CC(X)$ is equipped with the Mosco topology τ_M and $CB(X)$ is equipped with the Hausdorff metric topology induced by the norm of X, then* Cent: $CB(X) \times CC(X) \to CC(X)$ *is weakly usco.*

If F is a singleton set $\{x\}$, then $\mathrm{Cent}(F; A) = P(x, A)$, the metric projection of x onto A. Therefore, Corollary 3.7 includes [9, Theorem 5.1] as a special case. We mention that upper semicontinuity of the metric projection as a function of the set argument alone, topologized by Hausdorff distance, was considered apparently for the first time in [12].

For the remaining part of this section, we denote by $\mathrm{remote}(X)$ the collection of nonempty closed bounded subsets of X admitting farthest points. Nonempty compact sets and, more generally, nonempty M-compact sets [33], i.e., sets for which maximizing sequences admit convergent subsequences, are members of $\mathrm{remote}(X)$. The next theorem gives conditions for the relative center map to be (norm) usco.

THEOREM 3.8. *Suppose $X \in (H^*)$. If $C^*C(X^*)$ is equipped with the dual Mosco topology τ_{M^*} and $\mathrm{remote}(X^*)$ is equipped with the Hausdorff metric topology τ_H induced by the norm of X^*, then* $\mathrm{Cent}\colon \mathrm{remote}(X^*) \times C^*C(X^*) \to C^*C(X^*)$ *is usco.*

Proof. For (norm) upper semicontinuity, we proceed exactly as in the proof of Theorem 3.6 to prove that $\mathrm{Cent}^{-1}(E) = \{(F, A) \in \mathrm{remote}(X^*) \times C^*C(X^*) : \mathrm{Cent}(F; A) \cap E \neq \varnothing\}$ is closed for each *norm closed* subset E of X^*. We take a net $\langle (F_\lambda, A_\lambda) \rangle_{\lambda \in \Lambda}$ in $\mathrm{Cent}^{-1}(E)$ convergent to (F, A) and for each index λ, choose a point a_λ in $\mathrm{Cent}(F_\lambda; A_\lambda) \cap E$. Exactly as in the proof of Theorem 3.6, we may assume, by passing to a subnet if required, that $\langle a_\lambda \rangle$ w^*-converges to a point z, and that, necessarily, $z \in \mathrm{Cent}(F; A)$. Since $F \in \mathrm{remote}(X^*)$, there is a point $y \in F$ such that $\|z - y\| = r(F, z) = \mathrm{rad}(F; A)$. Exactly as in the proof of Theorem 3.6, we have for all λ sufficiently large,

$$\|z - y\| < \|a_\lambda - y\| + \varepsilon \leqslant r(F, a_\lambda) + \varepsilon \leqslant r(F_\lambda, a_\lambda) + 2\varepsilon$$
$$< \mathrm{rad}(F; A) + 3\varepsilon = \|z - y\| + 3\varepsilon.$$

Therefore, $\|a_\lambda - y\| \to \|z - y\|$. Also, since $\langle a_\lambda - y \rangle$ w^*-converges to $z - y$ and $X \in (H^*)$, $\langle a_\lambda \rangle$ converges to z in the norm topology. Since E is norm closed, we have $z \in \mathrm{Cent}(F; A) \cap E$ so that $(F, A) \in \mathrm{Cent}^{-1}(E)$.

Norm compactness of $\mathrm{Cent}(F; A)$ for each $F \in \mathrm{remote}(X^*)$ and each $A \in C^*C(X^*)$ follows from its w^*-compactness, using an almost identical argument as the one just given. The details are left to the reader. ∎

COROLLARY 3.9. *Suppose $X \in (Rf) \cap (H)$. If $CC(X)$ is equipped with the Mosco topology τ_M and $\mathrm{remote}(X)$ is equipped with the Hausdorff metric topology τ_H, then* $\mathrm{Cent}\colon \mathrm{remote}(X) \times CC(X) \to CC(X)$ *is usco.*

From [1, Lemma 1.2] it follows that if $X \in (R^*)$, then for each $A \in C^*C(X^*)$ and each $F \in \mathrm{remote}(X^*)$, the relative center $\mathrm{Cent}(F; A)$ is a singleton. In this case, Cent can be regarded as a mapping of remote$(X^*) \times C^*C(X^*)$ into X^*. In this context, Theorem 3.8 and Corollary 3.9 may be restated as follows:

COROLLARY 3.10. *Suppose* $X \in (R^*) \cap (H^*)$. *If* $C^*C(X^*)$ *is equipped with the topology* τ_{M^*} *and remote*(X^*) *is equipped with the topology* τ_H, *then* Cent: remote$(X^*) \times C^*C(X^*) \to X^*$ *is single valued and continuous.*

COROLLARY 3.11. *Suppose* $X \in (Rf) \cap (R) \cap (H)$. *If* $CC(X)$ *is equipped with the topology* τ_M *and remote*(X) *is equipped with the topology* τ_H, *then* Cent: remote$(X) \times CC(X) \to X$ *is single valued and continuous.*

THEOREM 3.12. *Suppose* $X \in (R^*) \cap (H^*)$. *Consider the following five statements for a net* $\langle A_\lambda \rangle_{\lambda \in \Lambda}$ *in* $C^*C(X^*)$:

(1) $A = \tau_{M^*}\text{-}\lim A_\lambda$;

(2) $\lim_\lambda \text{Cent}(F; A_\lambda) = \text{Cent}(F; A)$, *for every* $F \in \text{remote}(X^*)$;

(3) $\lim_\lambda \text{rad}(F; A_\lambda) = \text{rad}(F; A)$, *for every* $F \in \text{remote}(X^*)$;

(4) $\lim_\lambda d(y, A_\lambda) = d(y, A)$, *for every* $y \in X^*$;

(5) $\lim_\lambda P(y, A_\lambda) = P(y, A)$, *for every* $y \in X^*$.

We have $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$. *Moreover, if in addition, we assume either* $(*)$: X *is finite dimensional, or* $(**)$: $X \in (Rf) \cap (R) \cap (H)$, *then conditions* (1) *through* (5) *are equivalent.*

Proof. $(1) \Rightarrow (2)$. This follows immediately from Corollary 3.10.

$(2) \Rightarrow (3)$. For $F \in \text{remote}(X^*)$, let $a_\lambda = \text{Cent}(F; A_\lambda)$ and let $a = \text{Cent}(F; A)$. By condition (2), $\lim_\lambda \|a_\lambda - a\| = 0$, so that by Lemma 3.1,

$$\lim_\lambda \text{rad}(F; A_\lambda) = \lim_\lambda r(F, a_\lambda) = r(F, a) = \text{rad}(F; A).$$

$(3) \Rightarrow (4)$. Assuming (3), we have for each $y \in X^*$,

$$\lim_\lambda d(y, A_\lambda) = \lim_\lambda \text{rad}(\{y\}; A_\lambda) = \text{rad}(\{y\}; A) = d(y, A).$$

If X is finite dimensional, then X^* is finite dimensional (and reflexive), and a subbase for $\tau_{M^*} = \tau_M$ on X^* consists of all sets of the form V where V is an open subset of X^* and $(K^c)^+$ where K is a compact subset of X^*. Thus, τ_{M^*} reduces to the *Fell topology* [8, 19], also called the *topology of closed convergence* [26], induced by the norm topology on X^*. The equivalence of conditions (1), (4), and (5) follows immediately from [8, Lemma 2.1 and Theorem 3.1], with only the assumption (R^*). Thus, in the finite dimensional case, conditions (1) through (5) are equivalent, only assuming (R^*).

Finally, assume $X \in (Rf) \cap (R) \cap (H)$. To establish the equivalence of conditions (1) through (5), we prove $(5) \Rightarrow (1) \Rightarrow (4) \Rightarrow (5)$. Much of the discussion that follows is adapted from Sonntag [37].

$(5) \Rightarrow (1)$. Suppose $A \in V^-$ for some norm open subset V of X^*. Choose $y \in A \cap V$. Since $y = P(y, A) = \lim_\lambda P(y, A_\lambda)$, we have $A_\lambda \cap V \neq \emptyset$ eventually, i.e., $A_\lambda \in V^-$ eventually. Next, let K be a w^*-compact subset of X^* disjoint from A. To prove that A_λ is disjoint from K eventually, it would suffice to prove that if $A_\lambda \cap K \neq \emptyset$ frequently, then $A \cap K \neq \emptyset$. To this end, assume $\langle A_\mu \rangle$ is a subnet of $\langle A_\lambda \rangle$ such that for each μ, $A_\mu \cap K \neq \emptyset$. For each index μ, choose y_μ in $A_\mu \cap K$. By the weak*-compactness of K, by passing to a subnet, we may assume $\langle y_\mu \rangle$ is w^*-convergent to some point $y \in K$. We show $y \in A$.

For each index μ, let $a_\mu = P(y, A_\mu)$. By the fundamental dual characterization of best approximations (see, e.g., [25, Section 22] or [37, III.9]) for each μ, we have $\langle J^{-1}(y - a_\mu), y_\mu - a_\mu \rangle \leq 0$. Therefore,

$$\langle J^{-1}(y - a_\mu), y_\mu - y \rangle + \langle J^{-1}(y - a_\mu), y - a_\mu \rangle \leq 0,$$

which, by the definition of the duality mapping, yields

$$\langle J^{-1}(y - a_\mu), y_\mu - y \rangle + \|y - a_\mu\|^2 \leq 0. \qquad (\#)$$

By assumption (5), $\|y - P(y, A)\| = \lim_\mu \|y - a_\mu\|$, and since J^{-1} is norm norm continuous, we have $\lim_\mu J^{-1}(y - a_\mu) = J^{-1}(y - P(y, A))$. Furthermore, since $y = w^*\text{-}\lim y_\mu$ we obtain $\lim_\mu \langle J^{-1}(y - a_\mu), y_\mu - y \rangle = 0$. Taking the limit in Eq. (#) above, we get $\|y - P(y, A)\| \leq 0$. We conclude that $y \in A$ must hold, so that $A \cap K \neq \emptyset$. Thus, $A = \tau_{M^*}\text{-}\lim A_\lambda$.

$(1) \Rightarrow (4)$. This is a special case of Lemma 3.3.

$(4) \Rightarrow (5)$. For each index λ, let $f_\lambda = d(\cdot, A_\lambda)^2/2$ and let $f = d(\cdot, A)^2/2$. As above, let $a_\lambda = P(y, A_\lambda)$. Under assumption $(**)$, by a theorem of Sonntag [37, III.10], the functions f_λ and f have unique subgradients at y, i.e., are Gateaux differentiable at y, and their derivatives at y are given by

$$f'_\lambda(y) = J^{-1}(y - a_\lambda) \qquad (\lambda \in \Lambda)$$

$$f'(y) = J^{-1}(y - P(y, A)).$$

In particular, for each $z \in X^*$, we have

(i) $f_\lambda(z) \geq f_\lambda(y) + \langle J^{-1}(y - a_\lambda), z - y \rangle$

(ii) $f(z) \geq f(y) + \langle J^{-1}(y - P(y, A)), z - y \rangle$.

For simplicity, set $x_\lambda = J^{-1}(y - a_\lambda)$ and $x = J^{-1}(y - P(y, A))$. Since the duality mapping is norm preserving, we have by condition (4)

(iii) $\lim_\lambda \|x_\lambda\| = \lim_\lambda \|y - a_\lambda\| = \lim_\lambda d(y, A_\lambda) = d(y, A) = \|y - P(y, A)\| = \|x\|$.

We claim that $\lim_\lambda \|x_\lambda - x\| = 0$. It suffices to show that each subnet $\langle x_\mu \rangle$ of $\langle x_\lambda \rangle$ has in turn a subnet $\langle x_\beta \rangle$ satisfying $\lim_\beta \|x_\beta - x\| = 0$. By reflexivity and (iii), the net $\langle x_\mu \rangle$, which is norm bounded eventually, must have a weakly convergent subnet $\langle x_\beta \rangle$. In view of (i), (ii), and assumption (4), the Gateaux differentiability f ensures that the (weak) limit of this subnet can only be x. Also by (iii), we have $\lim_\beta \|x_\beta\| = \|x\|$. It now follows that $\lim_\beta \|x_\beta - x\| = 0$ because $X \in (H)$, establishing the claim.

Since we now know that $\lim_\lambda \|x_\lambda - x\| = 0$, the norm-norm bicontinuity of the duality map J ensures that

$$\lim_\lambda \|P(y, A_\lambda) - P(y, A)\| = \lim_\lambda \|(y - a_\lambda) - (y - P(y, A))\|$$
$$= \lim_\lambda \|x_\lambda - x\| = 0.$$

Thus (5) follows from (4), completing the proof of the equivalence of conditions (1) through (5) in case (**). ∎

COROLLARY 3.13. *Let X be a normed space. Suppose either* (*): *X is finite dimensional and in* (R), *or* (**): $X \in (F) \cap (F^*)$. *Then for a net* $\langle A_\lambda \rangle_{\lambda \in \Lambda}$ *in* $CC(X)$, *the following are equivalent*:

(1) $A = \tau_M\text{-}\lim A_\lambda$;

(2) $\lim_\lambda \mathrm{Cent}(F; A_\lambda) = \mathrm{Cent}(F; A)$, *for every* $F \in \mathrm{remote}(X)$;

(3) $\lim_\lambda \mathrm{rad}(F; A_\lambda) = \mathrm{rad}(F; A)$, *for every* $F \in \mathrm{remote}(X)$;

(4) $\lim_\lambda d(x, A_\lambda) = d(x, A)$, *for every* $x \in X$;

(5) $\lim_\lambda P(x, A_\lambda) = P(x, A)$, *for every* $x \in X$.

A number of historical remarks are in order. The equivalence of statements (1), (4), and (5) in Corollary 3.13 for a *sequence* $\langle A_n \rangle$ in $CC(X)$, for X a Hilbert space, is due to Attouch [5]. For $X \in (F) \cap (F^*)$, the equivalence of (1), (4), and (5), again for sequences, is established in the thesis of Sonntag [37], and we are in essence building on his ideas. An alternative proof in this setting based on Moreau Yosida approximation can be found in [6], and a third proof is given in [38].

4. BAIRE CATEGORY RESULTS FOR CLOSED CONVEX SETS

By [9, Theorem 4.3], when X is separable and reflexive, $CC(X)$ equipped with the Mosco topology τ_M is a Polish space (second countable and completely metrizable) and therefore, the product space $CB(X) \times CC(X)$ is complete if $CB(X)$ is equipped with the topology τ_H [26, p. 45] (note: $\langle CB(X), \tau_H \rangle$ is not separable unless X is finite dimensional). Since the product so topologized is a Baire space, it is meaningful to ask whether the following generic statement is true in the sense of Baire category: is $\mathrm{Cent}(F; A)$ single-valued for most (F, A) in $CB(X) \times CC(X)$? From the point of view of optimization theory, this is equivalent to asking: does the convex programming problem

$$\text{minimize } r(x, F)$$

$$\text{subject to } x \in A$$

have a unique solution for most problems (F, A)? The following example shows that the answer is negative. In fact, the set of problems with unique solutions need not even be dense in $CB(X) \times CC(X)$.

EXAMPLE. Let $X = R^2$, equipped with the box norm. For each $(\alpha_1, \alpha_2) \in R^2$, let $V(\alpha_1, \alpha_2) = \{(\beta_1, \beta_2): \|(\beta_1, \beta_2) - (\alpha_1, \alpha_2)\| < \frac{1}{10}\}$. Then

$$\mathscr{A} \equiv V(1, 1)^- \cap V(1, -1)^- \cap V(-1, 1)^- \cap V(-1, -1)^-$$

is τ_M-open in $CC(X)$, and if $A \in \mathscr{A}$, then $A \supset [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$. Now let $F_0 = \{(0, 1), (0, -1)\}$. We claim that for each (F, A) in the open set $\{F: H(F, F_0) < \frac{1}{10}\} \times \mathscr{A}$, $\mathrm{Cent}(F; A)$ contains more than one point. To see this, fix F with $H(F, F_0) < \frac{1}{10}$. If $\rho = \max\{\beta_2: (\beta_1, \beta_2) \in F\}$ and $\sigma = \min\{\beta_2: (\beta_1, \beta_2) \in F\}$, then

$$\{(x, (\rho + \sigma)/2): -\tfrac{1}{2} \leqslant x \leqslant \tfrac{1}{2}\} \subset \mathrm{Cent}(F; X).$$

Thus, for each $A \in \mathscr{A}$,

$$\{(x, (\rho + \sigma)/2): -\tfrac{1}{2} \leqslant x \leqslant \tfrac{1}{2}\} \subset \mathrm{Cent}(F; X) \cap A \subset \mathrm{Cent}(F; A).$$

What goes wrong here is that we allow constraint sets A to intersect $\mathrm{Cent}(F; X)$, the *absolute* center of F. We intend to show that our generic statement is true, provided we consider just those (F, A) for which $A \cap \mathrm{Cent}(F; X) = \varnothing$. In the sequel, for a real function f on X and $\alpha \in R$, denote by $\mathrm{sub}(f; \alpha)$ the *sublevel* set $\{x \in X: f(x) \leqslant \alpha\}$. The following facts, which we record as a lemma, are obvious and well known.

LEMMA 4.1. *Let f be a continuous convex function on a normed space X. For each real α, $\mathrm{sub}(f; \alpha)$ is a closed convex set, and for each $\alpha > \inf f$,*

$$\mathrm{int\,sub}(f; \alpha) = \{x \in X: f(x) < \alpha\}.$$

LEMMA 4.2. *Let X be a separable reflexive space. Suppose $CB(X)$ is equipped with the topology τ_H and $CC(X)$ is equipped with the topology τ_M. Then the subset $\Omega \equiv \{(F, A)\colon \mathrm{Cent}(F; X) \cap A = \varnothing\}$ of $CB(X) \times CC(X)$, equipped with the relative topology, is completely metrizable.*

Proof. We observe that

$$\Omega = \{(F, A) \in CB(X) \times CC(X)\colon \mathrm{rad}(F; A) - \mathrm{rad}(F; X) > 0\}.$$

By Lemma 3.4, the map $(F, A) \to \mathrm{rad}(F; A) - \mathrm{rad}(F; X)$ is continuous on $\langle CB(X), \tau_H\rangle \times \langle CC(X), \tau_M\rangle$. As a result, Ω is open in the product, and by a celebrated theorem of Alexandroff [41, p.179], any open subspace of a completely metrizable space is itself completely metrizable. ∎

The key ingredient in the proof of our generic theorem is a continuity theorem of Christensen [14], which may be viewed as a variant of the classical Kuratowski–Fort theorem [20]. Retaining the terminology of [9], we call a multifunction Γ from a topological space T to a normed space X *almost lower semicontinuous* (alsc) at $t_1 \in T$ if there exists $x_1 \in F(t_1)$ such that for each $\varepsilon > 0$, there exists a neighborhood V_ε of t_1 such that for each $t \in V_\varepsilon$, we have $\Gamma(t) \cap (x_1 + \varepsilon U) \neq \varnothing$. This property, considered first by Christensen under the name *Kenderov continuity*, agrees for compact valued multifunctions with a somewhat weaker continuity property introduced by Deutsch and Kenderov [16]. For our purposes, the following weakened form of Christensen's theorem suffices.

CHRISTENSEN'S THEOREM. *Let T be a complete metric space and let X be a Banach space. Suppose Γ is a weakly usco map from T to X. Then there exists a dense and G_δ subset G of T such that Γ is alsc at each $t \in G$.*

THEOREM 4.3. *Let X be a separable reflexive space. Suppose $CB(X)$ is equipped with the topology τ_H, $CC(X)$ is equipped with the topology τ_M, and the set $\Omega = \{(F, A) \in CB(X) \times CC(X)\colon \mathrm{Cent}(F; X) \cap A = \varnothing\}$ is equipped with the relative topology of $CB(X) \times CC(X)$. Then there exists a dense and G_δ subset Ω_0 of Ω such that for each (F, A) in Ω_0, $\mathrm{Cent}(F; A)$ is a singleton.*

Proof. By Corollary 3.7, $(F, A) \to \mathrm{Cent}(F; A)$ is weakly usco on $CB(X) \times CC(X)$, and thus on Ω. By Christensen's theorem there is a dense and G_δ subset Ω_0 of Ω such that at each (F, A) in Ω_0, the relative center map is alsc. Fix $(F_0, A_0) \in \Omega_0$. We claim that $\mathrm{Cent}(F_0, A_0)$ is a singleton.

Assume the contrary, and let $x_1 \in \mathrm{Cent}(F_0; A_0)$ be as guaranteed by the definition of almost lower semicontinuity. Let x_0 be a different point of $\mathrm{Cent}(F_0; A_0)$, and set $\varepsilon = \|x_1 - x_0\|/2$. By almost lower semicontinuity of the relative center map, there exists $\delta > 0$, open subsets $V_1, V_2, \ldots, V_n$ of X, and a weakly compact subset K of X such that

$$\Sigma \equiv \bigcap_{i=1}^{n} V_i^- \cap (K^c)^+$$

is a neighborhood of A_0, and whenever $H(F, F_0) < \delta$ and $A \in \Sigma$, then $\text{Cent}(F; A) \cap (x_1 + \varepsilon U) \neq \emptyset$. Let $\alpha = \text{rad}(F_0; A_0)$. By the definition of Ω, we have $\alpha > \text{rad}(F_0; X)$. Applying Lemma 4.1, $\text{sub}(r(F_0, \cdot); \alpha)$ is a closed convex set with nonempty interior $\{x \in X : r(F_0, x) < \alpha\}$, and A_0 does not meet the interior of $\text{sub}(r(F_0, \cdot); \alpha)$. By the separation theorem, there exists $y \neq \theta^*$ in X^* such that

$$\sup\{\langle x, y\rangle : x \in \text{sub}(r(F_0, \cdot); \alpha)\} \leqslant \beta \leqslant \inf\{\langle x, y\rangle : x \in A_0\}.$$

Since $\text{Cent}(F_0; A_0) = \text{sub}(r(F_0, \cdot); \alpha) \cap A_0$, we actually have $\langle x, y\rangle = \beta$ for each $x \in \text{Cent}(F_0, A_0)$. By reflexivity of X, there exists $u \in S(X)$ such that $\langle u, y\rangle = \|y\|$. Choose $a_i \in A_0 \cap V_i$ for $i = 1, 2, \ldots, n$. Since K is weakly compact and each V_i is norm open, and $\text{conv}\{x_0, x_1, a_1, a_2, \ldots, a_n\} \cap K = \emptyset$, there exists $\lambda > 0$ such that $a_i + \lambda u \in V_i$ for each $i \leqslant n$, and

$$A_1 \equiv \text{conv}\{x_0, x_1 + \lambda u, a_1 + \lambda u, \ldots, a_n + \lambda u\} \cap K = \emptyset.$$

Thus, $A_1 \in \Sigma$. Now $\text{rad}(F_0; A_1) \leqslant r(F_0, x_0) = \alpha$. Also, if $x \in A_1$ and $x \neq x_0$, then there exists p in $\text{conv}\{x_1 + \lambda u, a_1 + \lambda u, \ldots, a_n + \lambda u\}$ and $\mu \in (0, 1]$ such that $x = \mu p + (1 - \mu) x_0$. As a result,

$$\langle x, y\rangle = \mu\langle p, y\rangle + (1 - \mu)\langle x_0, y\rangle \geqslant \mu\beta + \mu\lambda \|y\| + (1 - \mu)\beta > \beta,$$

and therefore $r(F_0, x) > \alpha$. Thus, x_0 is the unique minimizer of $r(F_0, \cdot)$ in A_1 so that $\text{Cent}(F_0; A_1) = \{x_0\}$. Therefore, $\text{Cent}(F_0; A_1) \cap (x_1 + \varepsilon U) = \emptyset$, which contradicts the almost lower semicontinuity of the map $(F, A) \to \text{Cent}(F; A)$ at (F_0, A_0), and completes the proof. ∎

The proof of Theorem 4.3 actually establishes the following single variable result, which is perhaps more attractive from the point of view of approximation theory than is Theorem 4.3 (but perhaps less attractive from the point of view of optimization).

THEOREM 4.4. *Let X be a separable reflexive space and suppose $CC(X)$ is equipped with the topology τ_M. Then for each $F \in CB(X)$, $\text{Cent}(F; A)$ is single valued for most $A \in CC(X)$ for which $A \cap \text{Cent}(F; X) = \emptyset$.*

Note that when F is a singleton $\{x\}$, $\text{Cent}(\{x\}; X) = \{x\}$, and if A meets $\{x\}$, then already $\text{Cent}(\{x\}; A)$ is single valued. Thus, Theorem 4.4 says, in particular, that for each $x \in X$, $P(x, A)$ is a singleton for most $A \in CC(X)$, equipped with the Mosco topology. This fact was not formally observed in [9]. As is well known, if $X = R^2$ with the box norm, the statement is not true if we equip $CC(X)$ with the Hausdorff metric topology.

5. The Distal Topology

Let X be a normed space. As we mentioned in Section 2, the Wijsman topology τ_W on $CC(X)$ is the weakest topology for which the functionals $A \to d(x, A)$ are continuous on $CC(X)$, for each $x \in X$. When X is reflexive, this topology agrees with the usually stronger *ball topology* τ_B [7], which has as a subbase all sets of the form W^- where W is an open ball and $(B^c)^+$, where B is a closed ball [9, Theorem 3.5]. This fact remains valid in a dual space X^*, provided we work with $C^*C(X^*)$ rather than with all of $CC(X^*)$. It is natural to inquire whether the analogous weak topology on $C^*C(X^*)$, induced now by maps of the form $A \to \mathrm{rad}(F; A)$, admits a concrete presentation as a "hit-and-miss" topology as well. We resolve this question immediately.

Definition. Let X be a normed space. The *distal topology* τ_D on $CC(X)$ is the weakest topology on $CC(X)$ such that for each closed and bounded subset F of X, $A \to \mathrm{rad}(F; A)$ is continuous on $CC(X)$.

By the definition of the distal topology, for each $x \in X$, $A \to \mathrm{rad}(\{x\}; A)$, i.e., $A \to d(x, A)$, is τ_D-continuous on $CC(X)$, whence $\tau_D \supset \tau_W$.

Theorem 5.1. *Let X^* be a dual normed space. The distal topology τ_D on $C^*C(X^*)$ is generated by all sets of the form V^- where V is norm open, and $(B^c)^+$ where B is an intersection of a finite family of balls of a common radius.*

Proof. Let τ be the topology on $C^*C(X^*)$ generated by all sets of the form V^-, where V is norm open, and $(B^c)^+$ where B is an intersection of a finite family of balls of a common radius. We first show that for each fixed closed and bounded subset F of X that $A \to \mathrm{rad}(F; A)$ is τ-continuous. Fix $A_0 \in C^*C(X^*)$. Let $\varepsilon > 0$ and $a_0 \in \mathrm{Cent}(F; A_0)$ be arbitrary, and set $V = \{y : \| y - a_0 \| < \varepsilon\}$. Suppose $A \in V^-$; then there exists $a \in A \cap \{y : \| y - a_0 \| < \varepsilon\}$. By Lemma 3.1,

$$\mathrm{rad}(F; A) \leqslant r(F, a) < r(F, a_0) + \varepsilon = \mathrm{rad}(F; A_0) + \varepsilon.$$

This proves τ-upper semicontinuity of $A \to \mathrm{rad}(F; A)$ at $A = A_0$. Lower semicontinuity at A_0 obviously occurs if $\mathrm{rad}(F; A_0) = \mathrm{rad}(F; X^*)$.

Otherwise, let α be any scalar strictly between rad$(F; X^*)$ and rad$(F; A_0)$. We claim that there is a finite subset E of F such that

$$A_0 \cap \left(\bigcap_{y \in E} (y + \alpha U^*) \right) = \emptyset.$$

If not, $\{A_0 \cap (y + \alpha U^*) : y \in F\}$ would be a family of w^*-compact subsets of X^* with the finite intersection property. Thus, $\bigcap_{y \in F} A_0 \cap (y + \alpha U^*)$ would be nonempty, and choosing a point p in the intersection, we obtain

$$\text{rad}(F; A_0) \leqslant r(F, p) = \inf\{\beta : p \in y + \beta U^* \text{ for each } y \in F\} \leqslant \alpha.$$

This contradicts rad$(F; A_0) > \alpha$, establishing the claim. For such a finite set E, set $B = \bigcap_{y \in E} (y + \alpha U^*)$. By the definition of the toplogy τ, $(B^c)^+$ is τ-open. Clearly, $A_0 \in (B^c)^+$, and if $A \in (B^c)^+$, we have rad$(F; A) \geqslant$ rad$(E; A) > \alpha$. This establishes τ-lower semicontinuity of $A \to \text{rad}(F; A)$, and τ-continuity now follows. Thus, $\tau \supset \tau_D$.

It remains to show that $\tau \subset \tau_D$. Since $\tau_W \subset \tau_D$, we have $V^- \in \tau_D$ for each norm open subset V of X^*. Suppose B is a finite intersection of balls, say

$$B = \bigcap_{y \in E} (y + \alpha U^*),$$

where E is a finite subset of X^* and $\alpha > 0$. Now by the definition of the distal topology, $\{A \in C^*C(X^*) : \text{rad}(E; A) > \alpha)$ is τ_D-open. But this is precisely $(B^c)^+$, whence the distal topology contains τ. Thus, $\tau = \tau_D$. ∎

By the last theorem, it is clear that in a dual normed space, the dual Mosco topology τ_{M^*} includes the distal topology. As a result, we recover once again Lemma 3.3. At this point, we find it convenient to list the relationships between the many topologies we have introduced on $C^*C(X^*)$: The Hausdorff metric topology τ_H, the dual Mosco topology τ_{M^*}, the Wisjman topology τ_W, and the distal topology τ_D, the Fell topology τ_F, and the ball topology τ_B. The reader should easily be able to verify the inclusions below, mimicing arguments valid in the reflexive case [9, Theorem 3.5].

THEOREM 5.2. *Let X^* be a dual normed space. On $C^*C(X^*)$ we have*

$$\tau_F \subset \tau_W = \tau_B \subset \tau_D \subset \tau_{M^*} \subset \tau_H.$$

If $X \in (F) \cap (F^*)$, then $\tau_W = \tau_B = \tau_D = \tau_{M^*}$, and if X is finite dimensional, then $\tau_F = \tau_W = \tau_B = \tau_D = \tau_{M^*}$.

We next show that for a separable dual normed space X^*, the distal topology on $C^*C(X^*)$ is metrizable. To do this, it is sufficient to show, by the Urysohn metrization theorem [41, p. 166], that the topology is Hausdorff, second countable, and regular.

THEOREM 5.3. *Let X^* be a separable dual normed space. Then the distal topology τ_D on $C^*C(X^*)$ is second countable and metrizable.*

Proof. Suppose E is a countable dense subset of X^*. We claim that all sets of the form $(\{y: \|y-y_0\|<\alpha\})^-$ where $y_0 \in E$ and α is rational, and $(B^c)^+$, where B is a finite intersection of balls with centers in E, each of the same rational radius, is a (countable) subbase for τ_D. First, suppose V is a norm open subset of X^* and $A_0 \in V^-$. Choose $a_0 \in V \cap A_0$ and $\varepsilon > 0$ such that $\{y: \|y-a_0\|<\varepsilon\} \subset V$. There exists $e_0 \in E$ and a positive rational δ such that $a_0 \in \{y: \|y-e_0\|<\delta\} \subset \{y: \|y-a_0\|<\varepsilon\}$. As a result, $\{y: \|y-e_0\|<\delta\}^-$ is a τ_D-neighborhood of A_0 contained in V^-.

Next, let $\{y_1, y_2, y_3, ..., y_m\}$ be a finite subset of X^*, and let α be a positive scalar such that $B \equiv \bigcap_{i \leqslant m}(y_i + \alpha U^*)$ is nonempty. Suppose $A_0 \in (B^c)^+$. Evidently, $\text{rad}(\{y_1, y_2, ..., y_m\}; A_0) > \alpha$; so, there exists $n \in Z^+$ such that $\alpha + 3/n < \text{rad}(\{y_1, y_2, ..., y_m\}; A_0)$. Choose for each $i \in \{1, 2, ..., m\}$ a point $e_i \in E$ such that $\|e_i - y_i\| < 1/n$. Also, let β be a rational with $\alpha + 1/n < \beta < \alpha + 2/n$. Then $B_1 \equiv \bigcap_{i \leqslant m}(e_i + \beta U)$ contains B, and by Lemma 3.1

$$|\text{rad}(\{e_1, e_2, ..., e_m\}; A_0) - \text{rad}(\{y_1, y_2, ..., y_m\}; A_0)| \leqslant H(\{e_1, e_2, ..., e_m\}, \{y_1, y_2, ..., y_m\}) \leqslant 1/n.$$

Thus, by the choice of the index n,

$$\text{rad}(\{e_1, e_2, ..., e_m\}; A_0) \geqslant \alpha + 3/n - 1/n = \alpha + 2/n > \beta.$$

As a result, $A_0 \in (B_1^c)^+ \subset (B^c)^+$. The existence of a countable subbase, and thus a countable base, for τ_D is established.

It remains to show that τ_D is Hausdorff and regular. The proof that the Mosco topology τ_M is Hausdorff shows equally well that τ_D is Hausdorff [9, Theorem 3.4]. *Complete* regularity of τ_D follows from the general principle that a weak topology on a set determined by a family of real valued functions is automatically completely regular, because such a topology admits a compatible uniformity [41, p. 256]. Specifically, if T is a set and $\{f_i: i \in I\}$ is a family of real valued functions defined on T, then a base for a uniformity on T compatible with the weakest topology on T with respect to which each f_i is continuous consists of all subsets of $T \times T$ of the form

$$U[I_0; \varepsilon] = \{(t_1, t_2): |f_i(t_1) - f_i(t_2)| < \varepsilon \text{ for each } i \in I_0\},$$

where I_0 is a finite subset of I and ε is a positive real. ∎

Let X be a reflexive Banach space. In closing, we exhibit a curious duality between the distal and Mosco topologies on $CC(X)$ which might lead us to alternatively call the Mosco topology on $CC(X)$ the *proximal topology*. As a point of departure, note that "closed and bounded" may be replaced by "weakly compact and convex" in the above definition of the distal topology for a reflexive space, because for each closed and bounded subset F of X, we have $\text{rad}(F; \cdot) = \text{rad}(\text{cl conv } F; \cdot)$. Thus, for X reflexive, the distal topology is the weakest topology on $CC(X)$ such that for each weakly compact convex set K, the functional

$$A \to \inf\left\{\alpha : A \text{ meets } \bigcap_{x \in K} (x + \alpha U)\right\}$$

is continuous on $CC(X)$. The Mosco topology admits a dual description, replacing intersection by union. What is first required is a totally convex description of the Mosco topology, which should have been presented in [9], but was not.

LEMMA 5.4. *Let X be a reflexive Banach space. The Mosco topology τ_M on $CC(X)$ is generated by all sets of the form W^- where W is an open ball in X, and $(K^c)^+$ where K is a weakly compact convex subset of X.*

Proof. It suffices to show that each subbasic open set in the standard description of the Mosco topology is open in the topology τ generated by all sets of the form W^- where W is an open ball in X, and $(K^c)^+$ where K is a weakly compact and convex. Suppose $A \in V^-$ where V is an open subset of X. Choose $a_0 \in A \cap V$ and $\varepsilon > 0$ with $a_0 + \varepsilon U \subset V$. Then $W = \{x : \|x - x_0\| < \varepsilon\}$ is open ball, and $A \in W^- \subset V^-$. This shows that $V^- \in \tau$. On the other hand, suppose $A \cap K = \varnothing$ where K is weakly compact, but not necessarily convex. By the separation theorem, for each $k \in K$ there exists $y_k \in X^*$ and $\alpha_k \in R$ such that

$$\inf_{x \in A} \langle x, y_k \rangle > \alpha_k > \langle k, y_k \rangle.$$

Since $K \subset \bigcup_{k \in K} \{x \in X : \langle x, y_k \rangle < \alpha_k\}$ and K is weakly compact, there exists a finite subset F of K with $K \subset \bigcup_{k \in F} \{x \in X : \langle x, y_k \rangle \leqslant \alpha_k\}$. For each $k \in F$, let $H_k = \{x \in X : \langle x, y_k \rangle \leqslant \alpha_k\}$. Since $K = \bigcup_{k \in F} (K \cap H_k)$, we have

$$K \subset \bigcup_{k \in F} \text{cl conv}(K \cap H_k).$$

Since K is norm bounded, $\text{cl conv}(K \cap H_k)$ is a weakly compact convex set for each $k \in F$. Also, $A \cap H_k = \varnothing$ implies $A \cap [\text{cl conv}(K \cap H_k)] = \varnothing$. As a result,

$$A \in \bigcap_{k \in F} (\text{cl conv}(K \cap H_k)]^c)^+ \subset (K^c)^+.$$

This proves that $(K^c)^+$ is τ-open, and $\tau = \tau_M$. ∎

THEOREM 5.5. *Let X be a reflexive Banach space. The Mosco topology τ_M is the weakest topology on $CC(X)$ such that for each weakly compact convex set K, the functional*

$$A \to \inf\left\{\alpha : A \text{ meets } \bigcup_{x \in K} (x + \alpha U)\right\}$$

is continuous on $CC(X)$.

Proof. Let τ be the weak topology on $CC(X)$ so described. For each weakly compact convex set K and each closed convex set A, we have

$$\inf\left\{\alpha : A \text{ meets } \bigcup_{x \in K} (x + \alpha U)\right\} = \inf\{\|a - k\| : a \in A, k \in K\}.$$

Now it is known that the Mosco topology is the weakest one on $CC(X)$ such that $A \to \inf\{\|a-k\| : a \in A, k \in K\}$ is continuous for each weakly compact set K [9, Theorem 3.3]. Thus, $\tau \subset \tau_M$. But if K is a fixed weakly compact convex set, then

$$\{A \in CC(X) : \inf\{\|a-k\| : a \in A, k \in K\} > 0\} = (K^c)^+.$$

Thus, τ must contain $(K^c)^+$ for each weakly compact convex subset K of X. Also, letting K run over the singleton subsets of X, we see that τ must contain τ_W and thus V^- for each norm open set V. By the Lemma 5.4, $\tau = \tau_M$. ∎

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